

مهارات التفكير العليا

التكامل بالكسور الجزئية

تبرير: أحل السؤالين الآتيين تباعاً:

(33) أجد: $\int dx \sqrt{1+e^x}$ بطريقتين مختلفتين، إحداهما الكسور الجزئية، مبرراً أجابتي.

الحل الأول بضرب كل من البسط والمقام بـ e^{-x}

$$\int (e^{-x}+1)+C e^x dx = \int e^{-x} e^{-x} + 1 dx = -\int e^{-2x} + 1 dx = -\ln|1+e^{-x}| + C$$

الحل الثاني بالتعويض:

$$u = e^x \Rightarrow du = e^x dx = u dx \Rightarrow dx = \frac{du}{u} \int \sqrt{1+e^x} dx = \int \sqrt{1+u} \times \frac{du}{u} = \int \frac{\sqrt{1+u}}{u} du$$

$$\frac{\sqrt{1+u}}{u} = \frac{A}{u} + \frac{B}{\sqrt{1+u}} \Rightarrow 1 = A\sqrt{1+u} + Bu \Rightarrow A = -1, B = 1$$

$$\int \frac{\sqrt{1+u}}{u} du = \int \left(\frac{-1}{u} + \frac{1}{\sqrt{1+u}} \right) du = -\ln|u| + \ln|1+u| + C$$

$$= -\ln(e^x) + \ln(1+e^x) + C = \ln\left(\frac{1+e^x}{e^x}\right) + C = \ln\left(1 + \frac{1}{e^x}\right) + C$$

(34) أجد: $\int \frac{1}{1+e^x} dx$

$$\int \frac{1}{1+e^x} dx = \int \frac{1}{1+u} \frac{du}{u} = \int \frac{1}{u(1+u)} du = \int \left(\frac{A}{u} + \frac{B}{1+u} \right) du$$

$$1 = A(1+u) + Bu \Rightarrow A = 1, B = -1$$

$$\int \frac{1}{u(1+u)} du = \int \left(\frac{1}{u} - \frac{1}{1+u} \right) du = \ln|u| - \ln|1+u| + C = \ln\left|\frac{u}{1+u}\right| + C = \ln\left|\frac{e^x}{1+e^x}\right| + C$$

(35) تبرير: أثبت أن: $\int \frac{5x^2 - 8x + 12}{(x-1)^2} dx = \ln|3x-2| + \frac{1}{x-1} + C$

$$\frac{5x^2 - 8x + 12}{(x-1)^2} = \frac{A}{x-1} + \frac{B}{(x-1)^2} \Rightarrow 5x^2 - 8x + 12 = A(x-1) + B$$

$$5x^2 - 8x + 12 = Ax - A + B \Rightarrow A = 5, B = 17$$

$$\int \frac{5x^2 - 8x + 12}{(x-1)^2} dx = \int \left(\frac{5}{x-1} + \frac{17}{(x-1)^2} \right) dx = 5 \ln|x-1| - \frac{17}{x-1} + C$$

(36) تبرير: أثبت أن: $\int \frac{1}{(x^2+1)^2} dx = \frac{x}{2(x^2+1)} + \frac{1}{2} \arctan(x) + C$

$$u=x \Rightarrow u^2=x \Rightarrow dx=2u du \Rightarrow \int \frac{1}{x} dx = \int \frac{1}{u^2} \cdot 2u du = \int \frac{2}{u} du = 2 \ln|u| + C = 2 \ln|x| + C$$

(37) تبرير: أثبت أن: $\int \frac{1}{x^2+9x+4} dx = \frac{1}{2} \ln|x+1| - \frac{1}{2} \ln|x+3| + C$

$$\frac{1}{x^2+9x+4} = \frac{1}{(x+1)(x+3)} = \frac{A}{x+1} + \frac{B}{x+3} \Rightarrow 1 = A(x+3) + B(x+1)$$

$$1 = Ax + 3A + Bx + B = (A+B)x + (3A+B)$$

$$\begin{cases} A+B=0 \\ 3A+B=1 \end{cases} \Rightarrow \begin{matrix} A=-1 \\ B=1 \end{matrix}$$

$$\int \frac{1}{x^2+9x+4} dx = \int \left(\frac{-1}{x+1} + \frac{1}{x+3} \right) dx = -\ln|x+1| + \ln|x+3| + C = \ln\left| \frac{x+3}{x+1} \right| + C$$

تحذ: أجد كلاً من التكاملات الآتية:

(38) $\int \frac{1}{x^2+1} dx$

$$u=1+x \Rightarrow du=dx \Rightarrow \int \frac{1}{x^2+1} dx = \int \frac{1}{u^2-1} du = \int \frac{1}{(u-1)(u+1)} du$$

$$\frac{1}{(u-1)(u+1)} = \frac{A}{u-1} + \frac{B}{u+1} \Rightarrow 1 = A(u+1) + B(u-1)$$

$$1 = Au + A + Bu - B = (A+B)u + (A-B)$$

$$\begin{cases} A+B=0 \\ A-B=1 \end{cases} \Rightarrow \begin{matrix} A=-1/2 \\ B=1/2 \end{matrix}$$

$$\int \frac{1}{x^2+1} dx = \int \left(\frac{-1/2}{u-1} + \frac{1/2}{u+1} \right) du = -\frac{1}{2} \ln|u-1| + \frac{1}{2} \ln|u+1| + C = \frac{1}{2} \ln\left| \frac{u+1}{u-1} \right| + C = \frac{1}{2} \ln\left| \frac{x+2}{x} \right| + C$$

(39) $\int \frac{1}{x^4-1} dx$

$$\frac{1}{x^4-1} = \frac{1}{(x^2+1)(x-1)(x+1)} = \frac{A}{x^2+1} + \frac{B}{x-1} + \frac{C}{x+1}$$

$$1 = A(x-1)(x+1) + B(x^2+1)(x-1) + C(x^2+1)(x+1)$$

$$1 = A(x^2-1) + B(x^3-x^2-x+1) + C(x^3+x^2+x+1)$$

$$1 = (A+B+C)x^3 + (-A-B+C)x^2 + (A-B+C)x + (A+B+C)$$

$$\begin{cases} A+B+C=0 \\ -A-B+C=0 \\ A-B+C=1 \\ A+B+C=1 \end{cases} \Rightarrow \begin{matrix} A=-1/2 \\ B=1/2 \\ C=1 \end{matrix}$$

$$\int \frac{1}{x^4-1} dx = \int \left(\frac{-1/2}{x^2+1} + \frac{1/2}{x-1} + \frac{1}{x+1} \right) dx = -\frac{1}{2} \arctan\left(\frac{1}{x}\right) + \frac{1}{2} \ln|x-1| + \ln|x+1| + C$$

$$\int (2x+1)|2x-1| + 116 \ln(4x^2+1) + 116 \ln|2x-1| + 182x+1) dx = -116 \ln|4x^2-14x^2+1| + CC = 116 \ln$$

$$\int (1x-x^3) dx \quad (40)$$

$$u = x^6 \Rightarrow du = 6x^5 dx \Rightarrow dx = \frac{du}{6x^5} = \frac{du}{6u^{5/6}} = \frac{1}{6} u^{-5/6} du$$

$$\int (1x-x^3) dx = \int (u^{1/6} - u^{3/6}) \cdot \frac{1}{6} u^{-5/6} du = \frac{1}{6} \int (u^{1/6-5/6} - u^{3/6-5/6}) du = \frac{1}{6} \int (u^{-2/3} - u^{-2/6}) du$$

$$= \frac{1}{6} \left(\int u^{-2/3} du - \int u^{-1/3} du \right) = \frac{1}{6} \left(3u^{1/3} - 3u^{2/3} \right) + C = \frac{1}{2} (u^{1/3} - u^{2/3}) + C$$

$$= \frac{1}{2} (x^2 - x^4) + C$$