

مهارات التفكير العليا

التكامل بالكسور الجزئية

تبرير: أحل السؤالين الآتيين تباعاً:

(33) أجد: $\int dx \sqrt{1+e^x}$ بطريقتين مختلفتين، إحداهما الكسور الجزئية، مبرراً أجابتي.

الحل الأول بضرب كل من البسط والمقام بـ e^{-x}

$$\int (e^{-x}+1)+C e^x dx = \int e^{-x} e^{-x} + 1 dx = -\int e^{-2x} + 1 dx = -\ln|1+e^{-x}| + C$$

الحل الثاني بالتعويض:

$$u = e^x \Rightarrow du = e^x dx = u dx \Rightarrow dx = \frac{du}{u} \int \sqrt{1+e^x} dx = \int \sqrt{1+u} \times \frac{du}{u} = \int \frac{\sqrt{1+u}}{u} du$$

$$\frac{\sqrt{1+u}}{u} = \frac{A}{u} + \frac{B}{\sqrt{1+u}} \Rightarrow 1 = A\sqrt{1+u} + Bu \Rightarrow A = -1, B = 1$$

$$\int \frac{\sqrt{1+u}}{u} du = \int \left(\frac{-1}{u} + \frac{1}{\sqrt{1+u}} \right) du = -\ln|u| + \ln|1+u| + C$$

$$= -\ln(e^x) + \ln(1+e^x) + C = \ln\left(\frac{1+e^x}{e^x}\right) + C = \ln\left(1 + \frac{1}{e^x}\right) + C$$

(34) أجد: $\int \frac{1}{1+e^x} dx$

$$\int \frac{1}{1+e^x} dx = \int \frac{1}{1+u} \times \frac{du}{u} = \int \frac{1}{u(1+u)} du = \int \left(\frac{A}{u} + \frac{B}{1+u} \right) du$$

$$1 = A(1+u) + Bu \Rightarrow A = 1, B = -1$$

$$\int \frac{1}{u(1+u)} du = \int \left(\frac{1}{u} - \frac{1}{1+u} \right) du = \ln|u| - \ln|1+u| + C = \ln\left|\frac{u}{1+u}\right| + C = \ln\left|\frac{e^x}{1+e^x}\right| + C$$

(35) تبرير: أثبت أن: $\int \frac{5x^2 - 8x + 12}{(x-1)^2} dx = \ln|3x-2| + \frac{1}{x-1} + C$

$$\frac{5x^2 - 8x + 12}{(x-1)^2} = \frac{A}{x-1} + \frac{B}{(x-1)^2} \Rightarrow 5x^2 - 8x + 12 = A(x-1) + B$$

$$5x^2 - 8x + 12 = Ax - A + B \Rightarrow A = 5, B = 17$$

$$\int \frac{5x^2 - 8x + 12}{(x-1)^2} dx = \int \left(\frac{5}{x-1} + \frac{17}{(x-1)^2} \right) dx = 5 \ln|x-1| - \frac{17}{x-1} + C$$

(36) تبرير: أثبت أن: $\int \frac{1}{(1+x^2)^2} dx = \frac{x}{2(1+x^2)} + \frac{1}{2} \arctan(x) + C$

$$u=x \Rightarrow u^2=x \Rightarrow dx=2u du \Rightarrow \int \frac{1}{2u} du = \int \frac{1}{2\sqrt{x}} dx = \sqrt{x} + C$$

$$\int \frac{1}{x^2} dx = \int x^{-2} dx = \frac{x^{-1}}{-1} + C = -\frac{1}{x} + C$$

$$\int \frac{1}{x^3} dx = \int x^{-3} dx = \frac{x^{-2}}{-2} + C = -\frac{1}{2x^2} + C$$

$$\int \frac{1}{x^4} dx = \int x^{-4} dx = \frac{x^{-3}}{-3} + C = -\frac{1}{3x^3} + C$$

$$\int \frac{1}{x^5} dx = \int x^{-5} dx = \frac{x^{-4}}{-4} + C = -\frac{1}{4x^4} + C$$

$$\int \frac{1}{x^6} dx = \int x^{-6} dx = \frac{x^{-5}}{-5} + C = -\frac{1}{5x^5} + C$$

$$\int \frac{1}{x^7} dx = \int x^{-7} dx = \frac{x^{-6}}{-6} + C = -\frac{1}{6x^6} + C$$

$$\int \frac{1}{x^8} dx = \int x^{-8} dx = \frac{x^{-7}}{-7} + C = -\frac{1}{7x^7} + C$$

$$\int \frac{1}{x^9} dx = \int x^{-9} dx = \frac{x^{-8}}{-8} + C = -\frac{1}{8x^8} + C$$

$$\int \frac{1}{x^{10}} dx = \int x^{-10} dx = \frac{x^{-9}}{-9} + C = -\frac{1}{9x^9} + C$$

$$\int \frac{1}{x^{11}} dx = \int x^{-11} dx = \frac{x^{-10}}{-10} + C = -\frac{1}{10x^{10}} + C$$

$$\int \frac{1}{x^{12}} dx = \int x^{-12} dx = \frac{x^{-11}}{-11} + C = -\frac{1}{11x^{11}} + C$$

$$\int \frac{1}{x^{13}} dx = \int x^{-13} dx = \frac{x^{-12}}{-12} + C = -\frac{1}{12x^{12}} + C$$

$$\int \frac{1}{x^{14}} dx = \int x^{-14} dx = \frac{x^{-13}}{-13} + C = -\frac{1}{13x^{13}} + C$$

$$\int \frac{1}{x^{15}} dx = \int x^{-15} dx = \frac{x^{-14}}{-14} + C = -\frac{1}{14x^{14}} + C$$

$$\int \frac{1}{x^{16}} dx = \int x^{-16} dx = \frac{x^{-15}}{-15} + C = -\frac{1}{15x^{15}} + C$$

$$\int \frac{1}{x^{17}} dx = \int x^{-17} dx = \frac{x^{-16}}{-16} + C = -\frac{1}{16x^{16}} + C$$

$$\int \frac{1}{x^{18}} dx = \int x^{-18} dx = \frac{x^{-17}}{-17} + C = -\frac{1}{17x^{17}} + C$$

$$\int \frac{1}{x^{19}} dx = \int x^{-19} dx = \frac{x^{-18}}{-18} + C = -\frac{1}{18x^{18}} + C$$

$$\int \frac{1}{x^{20}} dx = \int x^{-20} dx = \frac{x^{-19}}{-19} + C = -\frac{1}{19x^{19}} + C$$

$$\int \frac{1}{x^{21}} dx = \int x^{-21} dx = \frac{x^{-20}}{-20} + C = -\frac{1}{20x^{20}} + C$$

$$\int \frac{1}{x^{22}} dx = \int x^{-22} dx = \frac{x^{-21}}{-21} + C = -\frac{1}{21x^{21}} + C$$

$$\int \frac{1}{x^{23}} dx = \int x^{-23} dx = \frac{x^{-22}}{-22} + C = -\frac{1}{22x^{22}} + C$$

$$\int \frac{1}{x^{24}} dx = \int x^{-24} dx = \frac{x^{-23}}{-23} + C = -\frac{1}{23x^{23}} + C$$

$$\int \frac{1}{x^{25}} dx = \int x^{-25} dx = \frac{x^{-24}}{-24} + C = -\frac{1}{24x^{24}} + C$$

$$\int \frac{1}{x^{26}} dx = \int x^{-26} dx = \frac{x^{-25}}{-25} + C = -\frac{1}{25x^{25}} + C$$

$$\int \frac{1}{x^{27}} dx = \int x^{-27} dx = \frac{x^{-26}}{-26} + C = -\frac{1}{26x^{26}} + C$$

$$\int \frac{1}{x^{28}} dx = \int x^{-28} dx = \frac{x^{-27}}{-27} + C = -\frac{1}{27x^{27}} + C$$

$$\int \frac{1}{x^{29}} dx = \int x^{-29} dx = \frac{x^{-28}}{-28} + C = -\frac{1}{28x^{28}} + C$$

$$\int \frac{1}{x^{30}} dx = \int x^{-30} dx = \frac{x^{-29}}{-29} + C = -\frac{1}{29x^{29}} + C$$

$$\int \frac{1}{x^{31}} dx = \int x^{-31} dx = \frac{x^{-30}}{-30} + C = -\frac{1}{30x^{30}} + C$$

$$\int \frac{1}{x^{32}} dx = \int x^{-32} dx = \frac{x^{-31}}{-31} + C = -\frac{1}{31x^{31}} + C$$

$$\int \frac{1}{x^{33}} dx = \int x^{-33} dx = \frac{x^{-32}}{-32} + C = -\frac{1}{32x^{32}} + C$$

$$\int \frac{1}{x^{34}} dx = \int x^{-34} dx = \frac{x^{-33}}{-33} + C = -\frac{1}{33x^{33}} + C$$

$$\int \frac{1}{x^{35}} dx = \int x^{-35} dx = \frac{x^{-34}}{-34} + C = -\frac{1}{34x^{34}} + C$$

$$\int \frac{1}{x^{36}} dx = \int x^{-36} dx = \frac{x^{-35}}{-35} + C = -\frac{1}{35x^{35}} + C$$

$$\int \frac{1}{x^{37}} dx = \int x^{-37} dx = \frac{x^{-36}}{-36} + C = -\frac{1}{36x^{36}} + C$$

$$\int \frac{1}{x^{38}} dx = \int x^{-38} dx = \frac{x^{-37}}{-37} + C = -\frac{1}{37x^{37}} + C$$

$$\int \frac{1}{x^{39}} dx = \int x^{-39} dx = \frac{x^{-38}}{-38} + C = -\frac{1}{38x^{38}} + C$$

$$\int \frac{1}{x^{40}} dx = \int x^{-40} dx = \frac{x^{-39}}{-39} + C = -\frac{1}{39x^{39}} + C$$

$$\int \frac{1}{x^{41}} dx = \int x^{-41} dx = \frac{x^{-40}}{-40} + C = -\frac{1}{40x^{40}} + C$$

$$\int \frac{1}{x^{42}} dx = \int x^{-42} dx = \frac{x^{-41}}{-41} + C = -\frac{1}{41x^{41}} + C$$

$$\int \frac{1}{x^{43}} dx = \int x^{-43} dx = \frac{x^{-42}}{-42} + C = -\frac{1}{42x^{42}} + C$$

$$\int \frac{1}{x^{44}} dx = \int x^{-44} dx = \frac{x^{-43}}{-43} + C = -\frac{1}{43x^{43}} + C$$

$$\int \frac{1}{x^{45}} dx = \int x^{-45} dx = \frac{x^{-44}}{-44} + C = -\frac{1}{44x^{44}} + C$$

$$\int \frac{1}{x^{46}} dx = \int x^{-46} dx = \frac{x^{-45}}{-45} + C = -\frac{1}{45x^{45}} + C$$

$$\int \frac{1}{x^{47}} dx = \int x^{-47} dx = \frac{x^{-46}}{-46} + C = -\frac{1}{46x^{46}} + C$$

$$\int \frac{1}{x^{48}} dx = \int x^{-48} dx = \frac{x^{-47}}{-47} + C = -\frac{1}{47x^{47}} + C$$

$$\int \frac{1}{x^{49}} dx = \int x^{-49} dx = \frac{x^{-48}}{-48} + C = -\frac{1}{48x^{48}} + C$$

$$\int \frac{1}{x^{50}} dx = \int x^{-50} dx = \frac{x^{-49}}{-49} + C = -\frac{1}{49x^{49}} + C$$

(37) تبرير: أثبت أن: $\int \frac{1}{x^2+9x+4} dx = \frac{1}{5} \ln \left| \frac{x+2}{2x+3} \right| + C$

$$\frac{1}{x^2+9x+4} = \frac{1}{(x+2)(2x+3)} = \frac{A}{x+2} + \frac{B}{2x+3}$$

$$1 = A(2x+3) + B(x+2)$$

$$1 = 2Ax + 3A + Bx + 2B$$

$$1 = (2A+B)x + (3A+2B)$$

$$\begin{cases} 2A+B=0 \\ 3A+2B=1 \end{cases} \Rightarrow \begin{cases} A=-\frac{1}{5} \\ B=\frac{2}{5} \end{cases}$$

$$\int \frac{1}{x^2+9x+4} dx = \int \left(\frac{-1/5}{x+2} + \frac{2/5}{2x+3} \right) dx = -\frac{1}{5} \ln|x+2| + \frac{1}{5} \ln|2x+3| + C = \frac{1}{5} \ln \left| \frac{2x+3}{x+2} \right| + C$$

تحذ: أجد كلاً من التكاملات الآتية:

(38) $\int \frac{1}{x^2+1} dx$

$$\int \frac{1}{x^2+1} dx = \int \frac{1}{u^2+1} du = \arctan(u) + C = \arctan(x) + C$$

(39) $\int \frac{1}{x^4-1} dx$

$$\frac{1}{x^4-1} = \frac{1}{(x^2+1)(x^2-1)} = \frac{1}{(x^2+1)(x-1)(x+1)} = \frac{A}{x-1} + \frac{B}{x+1} + \frac{C}{x^2+1}$$

$$1 = A(x+1)(x^2+1) + B(x-1)(x^2+1) + C(x-1)(x+1)$$

$$1 = A(x^3+x^2+x+1) + B(x^3-x^2-x+1) + C(x^2-1)$$

$$1 = (A+B)x^3 + (A-B+C)x^2 + (A-B)x + (A+B-C)$$

$$\begin{cases} A+B=0 \\ A-B+C=1 \\ A-B=0 \\ A+B-C=1 \end{cases} \Rightarrow \begin{cases} A=1/4 \\ B=-1/4 \\ C=1/2 \end{cases}$$

$$\int \frac{1}{x^4-1} dx = \frac{1}{4} \ln|x-1| - \frac{1}{4} \ln|x+1| + \frac{1}{2} \arctan\left(\frac{x}{1}\right) + C$$

$$\int (2x+1)|2x-1| + 116 \ln(4x^2+1) + 116 \ln|2x-1| + 182x+1) dx = -116 \ln|4x^2-14x^2+1| + CC = 116 \ln$$

$$\int (1x-x^3) dx \quad (40)$$

$$u = x^6 \Rightarrow du = 6x^5 dx \Rightarrow dx = \frac{du}{6x^5} \Rightarrow dx = \frac{du}{6u^{5/6}} \Rightarrow dx = \frac{1}{6} u^{-5/6} du$$

$$u = x^6 \Rightarrow x = u^{1/6} \Rightarrow x^3 = u^{1/2} \Rightarrow \int (1x-x^3) dx = \int (u^{1/6} - u^{1/2}) \cdot \frac{1}{6} u^{-5/6} du = \frac{1}{6} \int (u^{-2/3} - u^{1/3}) du$$

$$= \frac{1}{6} \left(-3u^{1/3} - \frac{3}{4} u^{4/3} \right) + C = -\frac{1}{2} u^{1/3} - \frac{1}{8} u^{4/3} + C$$

$$= -\frac{1}{2} x^2 - \frac{1}{8} x^8 + C$$